On the Series of Legendre.

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In recent communications to the Society, I have confined myself largely to the Theory of Fourier Series, partly because much seemed to me still to require doing in this subject, partly because I believed its thorough investigation to be the natural preparation for the study of other series of normal functions. It has, indeed, been known for some time that the behaviour of, for instance, series of Sturm-Liouville functions exactly corresponds to that of Fourier series. The introduction that I have recently made into Analysis of what I have called *restricted Fourier series* enables us to notably extend the range of such analogies. I propose in the present communication to illustrate this remark with reference to series of Legendre coefficients.

Whereas Fourier series may be said to be "naturally unrestricted," in virtue of the fact that the convergence of the integrated series to an integral necessarily involves the tendency towards zero of its own general term, so that the consideration of the more general type of series does not at once suggest itself, Legendre series may be said to come into being "restricted," even when the coefficients are expressible in what may be called the Fourier form by means of integrals involving Legendre's coefficients. In other words, such series correspond precisely to restricted Fourier series, instead of to ordinary Fourier series like the analogous series of Sturm-Liouville functions.

Among these restricted Fourier series, those which I distinguished as *of the first class*, namely, those whose first integrated series converge to integrals in some interval or intervals, play the most important part. With regard to such series I have been able to demonstrate, among others, two fundamental theorems:—

I. *If the typical term of the series tends to zero, in which case I distinguish the series as ordinary restricted Fourier series (R.F. series); these series behave precisely in the same way as ordinary Fourier series in the interval or intervals to which they are restricted.*

II. *If this condition is not fulfilled, then the properties of the series which coincide with those of ordinary Fourier series are those in which the summation is performed in the Cesàro manner, index unity; properties involving ordinary* *convergence, or ordinary upper and lower functions, have in this case to be excluded.*

These theorems have an immediate application to Legendre series, and the mode of application is more general than might be at first supposed. It is, in fact, quite unnecessary to postulate in our theorems that the coefficients of the Legendre series should be expressible in the Fourier form. In other words, if we denote the Legendre series by

$$\sum_{n=0}^{\infty} a_n \mathrm{P}_n(\cos\theta), \tag{1}$$

then it is *not* necessary for the validity of our theorems that the "Fourier" relation

$$a_n = \frac{2n+1}{2}\int_0^{\pi} f(\cos\theta)\,\mathrm{P}_n(\cos\theta)\sin n\theta\, d\theta \tag{2}$$

should hold.

When (2) does hold, it is known that the series (1), when integrated term-by-term, converges to an integral throughout its interval of periodicity. The converse also is true, but this is not the case when all that we require is that the integrated series of (1) should converge to an integral *in one or more sub-intervals.*

It seems worth while to particularly call the attention of the Society to the fact, firstly, that the larger class of Legendre series, so obtained, possess, as regards the internal points of all such intervals, all the properties of the smaller class for which (2) holds: this follows immediately from our theory. As already pointed out, however, even if (2) holds, the series is, so to speak, naturally restricted owing to the failure of $a_n\mathrm{P}_n(\cos\theta)$ to tend to zero. This fact has certainly been responsible for the isolated character of the results hitherto obtained with respect to these series. For their satisfactory discussion, the previous consideration of restricted Fourier series is absolutely essential.

In a recent communication to the Society, I gave actual examples of restricted Fourier series of the ordinary type (R.F. series) which converge in the ordinary way, thus justifying from the practical standpoint their introduction. It will be gathered from what has been said, however, that their importance in theory is even greater than that in practice.

We are, indeed, able to completely dispose of the theory of series of Legendre coefficients, in virtue of the properties of the trigonometrical series above referred to.

In fact, by means of the asymptotic expansion of $\mathrm{P}_n(x)$ we are able to express a Legendre series, when multiplied by a suitable function, as the sum of a restricted Fourier series and a series of terms which converges uniformly in any closed interval which excludes the end-points of the interval of periodicity.

We thus at once obtain the theorem that, at an internal point of an interval to which such a series is restricted, equally whether this be the whole interval of periodicity of $P_n(\cos\theta)$ or not, the conditions of convergence of Legendre series are *the same* as those of an ordinary Fourier series, provided only, either the summation be effected in the Cesàro manner, index unity, in both series, or

$$a_n P_n(\cos\theta) \to 0, \qquad (n \to \infty),$$

which is the same thing as

$$n^{-\frac{1}{2}} a_n \to 0, \qquad (n \to \infty),$$

in which latter case, equally whether the summation employed be ordinary or Cesàro, the conditions are the same. Moreover, this is true with regard to uniform convergence, as well as with regard to convergence *per se.* We have, be it understood, to suppose that, if the coefficients a_n do not satisfy condition (2), the integrated series converges to an integral in the interval, or intervals, of restriction. Corresponding theorems, however, hold good when, without this being the case, we can still assert that, for some integer p, the p-th integrated series converges to a p-ple integral in each such interval. This is in accordance with what might be expected by reference to the higher class of restricted Fourier series.

It will be remarked that this method not only reduces the discussion of Legendre series to that of Fourier series, of a more or less restricted type, but that it does so without the intervention of any properties of Legendre coefficients, except those involved in the asymptotic expansion above referred to.

Further, the considerable analytical difficulties which arise when, for example, *fractional* Cesàro summation is employed, are entirely avoided. The work has been done once for all in discussing Fourier series; nothing remains but to interpret these results in the light of our fundamental theorems.

It must not be forgotten, however, that these remarks only relate to *internal* points of the whole interval of periodicity. When the extremities are considered, real singular points as they are for a Legendre series, even when it satisfies condition (2), the character of the discussion at once changes. I may, however, remark, *en passant,* for the information of Fellows of the Society, that in this case also the problem may be reduced to one involving the Fourier series of an auxiliary function, so that the results of the theory of these latter series can be again utilised.

It is interesting to remark, in view of the practical applications, that this theory has an important bearing on the question of the term-by-term integration of Legendre series when multiplied by another function provided we remain in the interior of the interval of periodicity, or of the interval of restriction, of the series. The theorems which in such cases hold for Fourier series hold for Legendre series.

With regard to this latter class of theorems, it is evident that the restriction which excludes the end-points of the interval is essential, owing to the fact that the theorems which hold good for the convergence at the extremities are not the same as at an internal point, even in the case when (2) is satisfied. Speaking generally, we may say that, with certain exceptions of a notable character, the theorems relating to the end-points of the interval are obtainable from those relating to the internal points of the interval, by replacing ordinary convergence by Cesàro convergence, or by increasing the index of the Cesàro convergence. This is true accordingly in general for integration theorems involving the whole closed interval, as compared with those involving only a portion of the interval. Yet this complication is unnecessary, in that one of these theorems is deducible from the Theorem of Parsival for a series of normal functions.

In view of the great importance of these integration theorems in practice, I propose to make a further communication with regard to them on a subsequent occasion.

It should be scarcely necessary to add that the considerations here exposed apply, *mutatis mutandis,* to other series of normal functions, provided these normal functions are expressible, when multiplied by suitable functions, in a similar manner to the coefficients of Legendre.